

PARTIAL ORDERS ON THE TYPES IN βN

BY
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Abstract. Three partial orders on the types of points in βN are defined and studied in this paper. Their relation to the types of points in $\beta N - N$ is also described.

Several natural partial orders can be given to the types of points in βN . The purpose of this paper is to give some of these orders wider publicity. I feel these orders are fundamental in the study of ultrafilters on the integers. I had hoped these orders would lead to a classification of the types of points in N^* . I no longer feel this is true, but connections with this important unsolved problem are discussed.

I. Let N denote the set of all positive integers and S the set of all subsets of N . Let βN denote the set of all ultrafilters on N and N^* the set of all free ultrafilters on N . For $M \subset N$ let $W(M)$ be the set of all terms of βN to which M belongs. Then the set of all $W(M)$ for $M \subset N$ forms a basis for a topology on βN and the resulting space is topologically the Čech compactification of the integers and N^* is topologically $\beta N - N$. To avoid ambiguity let n' be the ultrafilter to which the integer n belongs and N' the set of all fixed ultrafilters; thus $N^* = \beta N - N'$.

If p and q are points of a topological space X , p and q are of the same type in X provided there is a homeomorphism of X onto itself taking p into q . It is easy to see [1] that two ultrafilters on N are of the same type in βN if and only if there is a permutation of N which takes the members of one onto the members of the other. That Ω and θ are of the same type in βN will be denoted by $\Omega \sim \theta$ and $[\Omega]$ will denote the set of all ultrafilters on N which are of the same type as Ω . Clearly $\sim$ is an equivalence relation and $[\Omega]$ has c members.

The problem of characterizing the types of points in N^* is the problem of finding reasonable necessary and sufficient conditions on terms Ω and θ of N^* so that one can construct a permutation of S which preserves *infinite* intersections and takes the members of Ω onto the members of θ .

A term Ω of N^* is called a P -point provided, for every countable subcollection $\{E_n\}_{n \in N}$ of Ω , there is a term E of Ω such that $E - E_n$ is finite for all n . In [1] Walter Rudin proves that the continuum hypothesis [CH] implies the existence of P -points in N^* and that all P -points are of the same type in N^* . Booth [3] has shown, using Martin's axiom rather than [CH], that there are P -points in N^* without an $\aleph_1$ base.

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In the light of these results, classification of the types in N^* seems hopeless without some set theoretic assumptions. The results of this paper frequently use [CH] and the strong structure of P -points this implies. The required background reading is [1].

II. A sequence $\{\rho_n\}_{n \in N}$ of terms of βN is called discrete if there exists a sequence $\{E_n\}_{n \in N}$ of disjoint subsets of N such that $E_n \in \rho_j$ if and only if $n=j$. Let D be the set of all such discrete countable sequences of terms of βN . If $X = \{\rho_n\}_{n \in N} \in D$ and $\theta \in \beta N$, define $\theta_X = \{M \subset N \mid \{n \mid M \in \rho_n\} \in \theta\}$. That is θ_X is the image of θ under the natural homeomorphism of βN onto the closure of X which takes n to ρ_n . Observe that, if X belongs to D and Ω to the closure of X , there is a unique θ such that $\theta_X = \Omega$.

In [2] Z. Frolik says θ produces θ_X . Then he proves that $\theta \in \beta N$ implies that θ produces 2^c terms of βN but is produced by at most c terms. Frolik also observes that if $X \in D$ and $X \subset N^*$, then there are at most c terms of $\bar{X}$ which have the same type in N^* and hence there are 2^c types of points in N^* .

A. Let us prove that Frolik's producing relation is a partial ordering of the types in βN .

1. $\theta \sim \Omega$ implies that θ produces Ω . For, if π is a permutation of N such that $M \in \Omega$ if and only if $\pi(M) \in \theta$, then $X = \{\pi(n)\}_{n \in N}$ is such that $\theta_X = \Omega$.

2. If ϕ produces θ and θ produces Ω , then ϕ produces Ω . Suppose $X \in D$ and $\{\rho_n\}_{n \in N} = Y \in D$ and $\phi_Y = \theta$ and $\theta_X = \Omega$. For each $n \in N$ define $\mu_n = (\rho_n)_X$ and let $Z = \{\mu_n\}_{n \in N}$. Then $\Omega = \phi_Z$.

3. Suppose Ω produces θ and θ produces Ω . Then $\theta \sim \Omega$. Suppose $X = \{\eta_n\}_{n \in N} \in D$ and $Y = \{\rho_n\}_{n \in N} \in D$ and $\theta_X = \Omega$ and $\Omega_Y = \theta$. For $n \in N$, define $\mu_n = (\rho_n)_X$ and let $Z = \{\mu_n\}_{n \in N}$. As in 2, $M \in \Omega$ if and only if $\{n \mid M \in \mu_n\} \in \Omega$. Let $\{E_n\}_{n \in N}$ be a set of disjoint subsets of N with $E_n \in \mu_n$. Define a two-valued function $f: N \rightarrow \{0, 1\}$ as follows. Define $f(1) = 0$ and, if $n \in E_1$ and $n > 1$, define $f(n) = 1$. Assume $i > 1$ and $f(n)$ has been defined for all $n < i$ and all $n \in E_j$ where $j < i$. If $f(i)$ has been defined as 1 and $n > i$ and $n \in E_i$, define $f(n) = 0$. If $f(i)$ has not been defined as 1, define $f(i) = 0$ and, if $n > i$ and $n \in E_i$, define $f(n) = 1$. Exactly one of $f^{-1}(0)$ and $f^{-1}(1)$ belongs to Ω . Suppose $f^{-1}(0) \in \Omega$. Let $M = f^{-1}(0) \cap \{n \mid f^{-1}(0) \in \mu_n\}$; $M \in \Omega$. If $n \in M$, then the finite set $f^{-1}(0) \cap E_n \in \mu_n$; hence μ_n is not free. But μ_n is not free implies $\mu_n \in X$, and we can define $\pi: M \rightarrow N$ by $\mu_n = \eta_{\pi(n)}$. We can find $E \subset M$ such that $E \in \Omega$, $N - E$ is infinite, and $N - \pi(E)$ is infinite. Then π/E can be extended to a permutation p of N onto N . It is easy to check that $B \in \Omega$ if and only if $p(B) \in \theta$. Thus $\theta \sim \Omega$.

Now for Ω and θ in βN , define $[\theta] \leq [\Omega]$ if θ produces Ω . By 1, 2, and 3, $\leq$ is a partial order on the types in βN .

B. If $\Omega \in \beta N$, the set of all types in βN which precede $[\Omega]$ in $\leq$ is totally ordered by $\leq$. For suppose $[\phi] \leq [\Omega]$ and $[\theta] \leq [\Omega]$. There is $X = \{\rho_n\}_{n \in N} \in D$ and $Y \in D$ such that $\phi_Y = \theta_X = \Omega$. Temporarily ignore the order of terms of D and just use them as sets with closures in βN . And use $\bar{X}$ for the closure of X in βN . We need the

fact [4] that if $V \in D$ and $\Omega \in \beta N$ and $\Omega \in \bar{V}$ and $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, then Ω belongs to one and only one of $\bar{V}_1$ and $\bar{V}_2$. In our case $\Omega \in \bar{X}$ and $X = (X \cap \bar{Y}) \cup (X - \bar{Y})$ and $\Omega \in \bar{Y}$ and $Y = (Y \cap \bar{X}) \cup (Y - \bar{X})$. Since $V = (X - \bar{Y}) \cup (Y - \bar{X})$ is countable and discrete, using the fact again we have that Ω belongs to the closure of $X \cap \bar{Y}$ or $Y \cap \bar{X}$. Say Ω belongs to the closure of $X \cap \bar{Y}$. Let $L = \{n \mid \rho_n \in \bar{Y}\}$; then $L \in \theta$. For $n \in L$, define μ_n to be the unique term of βN such that $(\mu_n)_Y = \rho_n$; if $n \in N - L$ define $\mu_n = n'$. Then $Z = \{\mu_n\}_{n \in N} \in D$ and $\phi_Z = \theta$.

C. The Frolik order $\leq$ has been studied extensively and we have a great deal of information. The fixed ultrafilters form a type in βN which precedes all other types. In N^* , types of P -points are obviously minimal in this order. For $X \in \beta N$, let $X^* = \bar{X} - X$. K. Kunen [5] has shown that [CH] there is a non- P -point in N^* not in X^* for any countable subset X of N^* and [CH] there is a countable subset X of N^* and a point of X^* which is not in Y^* for any Y in D . Types of both of these points are clearly minimal in N^* under $\leq$. By B the order is tree-like and knowing that any term is preceded by at most c terms but followed by 2^c terms gives us the picture of a fast branching tree.

In his thesis [3] Booth defines the product $[\theta] \cdot [\Omega]$ for Ω and θ in βN to be $[\theta_X]$ where $X = \{\rho_n\}_{n \in N} \in D$ and $\rho_n \sim \Omega$. Obviously $[\theta] \leq [\theta] \cdot [\Omega]$ and it is easy to show that $[\theta] \cdot [\Omega]$ is well defined. For $n > 1$, $[\theta]^n$ is defined inductively as $[\theta] \cdot [\theta]^{n-1}$; then $[\theta]^a \cdot [\theta]^b = [\theta]^{a+b}$. Among other things Booth uses these definitions to construct infinite well-ordered increasing and well-ordered decreasing sequences in $\leq$.

If Ω and θ are of the same type in N^* , then the types which precede $[\Omega]$ by $\leq$ must be precisely those which precede $[\theta]$ by $\leq$. In [4] I showed that the condition is also sufficient if both Ω and θ are limit points of countable sets of P -points [CH]. But by Kunen's examples there are minimal elements $[\theta]$ and $[\phi]$ in N^* such that θ and ϕ are of different types in N^* . So the condition is clearly not sufficient. Also if Λ is any member of N^* , it is easy to show, using the methods of B, that $[\Lambda]$ is maximal in the set of predecessors of both $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$. Hence the predecessors of $[\Lambda] \cdot [\theta]$ and $[\Lambda] \cdot [\phi]$ are exactly the same but none of their terms have the same type in N^* .

III. A more general partial order on the types of βN will now be discussed. Let F be the set of all functions from N onto N . If $\Omega \in \beta N$ and $f \in F$, define $f(\Omega) = \{f(M) \mid M \in \Omega\}$.

A. Let us prove that F induces a partial order on the types of points in βN .

1. If Ω and θ belong to βN and $\Omega \sim \theta$, then there is an $f \in F$ such that $f(\Omega) = \theta$. For, by definition, there is a permutation f of N such that $M \in \Omega$ if and only if $f(M) \in \theta$.

2. If Ω, ϕ and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \phi$, then $g \circ f(\Omega) = \phi$.

3. Suppose that Ω and θ belong to βN and f and g to F and $f(\Omega) = \theta$ and $g(\theta) = \Omega$. We prove $\Omega \sim \theta$. Let $L = \{n \mid (g \circ f)(n) > n\}$, $M = \{n \mid (g \circ f)(n) = n\}$ and

$Q = \{n \mid (g \circ f)(n) < n\}$. Suppose $M \in \Omega$; then f/M is one-to-one. There is a subset E of M belonging to Ω such that $N - E$ and $N - f(E)$ are infinite, and f/E can be extended to a permutation p of N onto N . Thus $B \in \Omega$ if and only if $p(B) \in \theta$ and $\Omega \sim \theta$.

If $M \notin \Omega$ one of L and Q must belong to Ω . Suppose $L \in \Omega$. If n and k belong to L , let us say nek provided that, for some nonnegative integers i and j , $(g \circ f)^i(n) = (g \circ f)^j(k)$ where $(g \circ f)^0$ is the identity map. Clearly e is an equivalence relation. Let E be the set of all equivalence classes of subsets of L related by e . From each $A \in E$ select $A_0 \in A$. Then define a two-valued function $t: L \rightarrow \{0, 1\}$ as follows. If $n \in A \in E$ and $(g \circ f)^i(n) = (g \circ f)^j(A_0)$ then define $t(n)$ as 0 if $|i - j|$ is even and 1 if $|i - j|$ is odd. The function t is well defined and only one of $t^{-1}(0)$ and $t^{-1}(1)$ belongs to Ω . Suppose $t^{-1}(0) \in \Omega$. By 2, $(g \circ f)(t^{-1}(0)) \in \Omega$. But

$$(g \circ f)(t^{-1}(0)) \subset t^{-1}(1)$$

and this is a contradiction.

As before, if Ω and θ belong to βN , define $[\Omega] \geq [\theta]$ provided there is an $f \in F$ such that $f(\Omega) = \theta$. By 1, 2, and 3, $\geq$ is a partial order on the types of points in βN .

B. We make several very simple observations.

1. If Ω and θ belong to βN and $[\theta] \leq [\Omega]$, then $[\Omega] \geq [\theta]$. For $[\theta] \leq [\Omega]$ implies there is $X = \{\rho_n\}_{n \in N} \in D$ such that $\theta_X = \Omega$. And $X \in D$ implies there is a set $\{E_n\}_{n \in N}$ of disjoint subsets of N such that $E_n \in \rho_n$. If $n > 1$ and $i \in E_n$, define $f(i) = n$; and if $i \notin E_n$ for any $n > 1$, define $f(i) = 1$. Then $f \in F$ and $f(\Omega) = \theta$.

2. If $\theta \in \beta N$, in $\geq [\theta]$ is greater than at most c types but less than 2^c types. The first follows from the cardinality of F being c . The second follows from 1 and Frolik's result in §II. In fact using Frolik's proof one shows that if both $\theta \in \beta N$ and $f \in F$ are given and, for all n , $f^{-1}(n)$ is infinite, there are 2^c terms Ω of βN such that $f(\Omega) = \theta$. By contrast recall that if $\theta \in \beta N$ and $X \in D$ are given there is a unique Ω such that $\theta_X = \Omega$.

3. If θ and ϕ belong to βN , there is an Ω in βN such that $[\Omega] \geq [\theta]$ and $[\Omega] \geq [\phi]$. Select $f \in F$ such that $f^{-1}(i)$ is infinite for each $i \in N$. Now select $g \in F$ such that, for each i and j in N , $g^{-1}(j) \cap f^{-1}(i)$ is infinite. Then select $\rho_{ij} \in N^*$ such that $g^{-1}(j) \cap f^{-1}(i) \in \rho_{ij}$. For $i \in N$ define $\rho_i = \phi_{X_i}$ where $X_i = \{\rho_{ij}\}_{j \in N}$; and for $X = \{\rho_i\}_{i \in N}$ let $\Omega = \theta_X$. Then $f(\Omega) = \theta$ and $g(\Omega) = \phi$.

4. Suppose θ and ϕ belong to βN . Let $B = \{[\Omega] \mid [\Omega] \geq [\theta] \text{ and } [\Omega] \geq [\phi]\}$. Then $[\Omega] \in B$ is minimal in B if and only if, for all f and g in F such that $f(\Omega) = \theta$ and $g(\Omega) = \phi$, there is an $M \in \Omega$ such that for i and j in N , $f^{-1}(i) \cap g^{-1}(j) \cap M$ is at most a singleton. To prove the only if, suppose f and g are given and let $Q = \{(i, j) \in N \times N \mid f^{-1}(i) \cap g^{-1}(j) \neq \emptyset\}$. If Q is finite, $[\theta]$ and $[\phi]$ and $[\Omega]$ are N' ; if $\Omega = m'$, $M = \{m\}$ has the desired properties. If Q is infinite there is a one-to-one function q from N onto Q ; define $k \in F$ by $k^{-1}(n) = f^{-1}(i) \cap g^{-1}(j)$ where $(i, j) = q(n)$. Hence if, for $n \in N$, we define $f^*(n) = f(k^{-1}(n))$ and $g^*(n) = g(k^{-1}(n))$, then f^* and g^* belong to F and $f^*(k(\Omega)) = \theta$ and $g^*(k(\Omega)) = \phi$. But $k(\Omega) \sim \Omega$ only if there

is an $M \in \Omega$ such that k/M is one-to-one. Now to prove *if*, suppose f, g and h belong to F and $f(h(\Omega)) = \theta$ and $g(h(\Omega)) = \phi$. Then $f \circ h(\Omega) = \theta$ and $g \circ h(\Omega) = \phi$. So assume also that there is an M such that, for all i and j in N ,

$$(f \circ h)^{-1}(i) \cap (g \circ h)^{-1}(j) \cap M$$

is at most a singleton. Then h/M is one-to-one and thus $h(\Omega) \sim \Omega$.

5. K. Kunen has a beautiful proof [5] that $\succeq$ is not a total order. The same proof shows that there are c types no pair of which are ordered. And using the continuum hypothesis it is easy to show that there are 2^c pairwise unordered types in $\succeq$, even 2^c minimal in N^* types.

6. Together 3 and 5 imply that, unlike II B, this order is not treelike. That is, there are types in βN whose predecessors are not totally ordered by $\succeq$. In fact $\succeq$ is more rootlike; that is, things get together near the top.

C. In addition to the facts in B, what can we say about $\succeq$? Again N' , the type of all fixed ultrafilters, is less than all other types. If $[\Omega]$ is minimal in N^* under $\succeq$, then Ω is a P -point by definition. If θ is a P -point and $[\theta] \succeq [\phi]$, then ϕ is a P -point or a fixed ultrafilter. It is not hard to prove [CH] that there are types which are minimal in N^* . This was first proved by J. Keisler [6] and will be a corollary of the example given in IV C. We prove [CH] that above every P -point type is another P -point type. Thus two types of the same type in N^* may or may not be ordered under $\succeq$; for two P -point types which are minimal under $\succeq$ are not ordered.

Suppose $\theta \in \beta N$ is a P -point. Then [CH] there is a P -point Ω such that $[\Omega] \succeq [\theta]$ but $[\Omega] \neq [\theta]$.

Proof. Clearly [CH] implies that both F and S have cardinality $\aleph_1$; hence let $F = \{f_\alpha\}_{\alpha < \omega_1}$ and $S = \{S_\alpha\}_{\alpha < \omega_1}$. Let f be a term of F such that, for each n , $f^{-1}(n)$ has n terms. We build Ω so that $f(\Omega) = \theta$ by induction on the countable ordinals. Let $\mathcal{A}$ be the set of all $A \subset N$ such that for some $a \in N$ and all $n \in N$, the number of terms of $f^{-1}(n) \cap A$ is less than a ; observe that $\mathcal{A}$ is closed under finite union.

For each $\alpha \in \omega_1$, we define a countable subset Ω_α of subsets of N such that

- (1) For $\beta < \alpha$, $\Omega_\beta \subset \Omega_\alpha$. Also Ω_α is closed under finite intersection.
- (2) If $\alpha = \beta + 1$, there is a term X of Ω_α such that (a) $X \subset S_\beta$ or $X \subset N - S_\beta$ and (b) for some $n \in N$, $X \subset f_\beta^{-1}(n)$, or, for all $n \in N$, $X \cap f_\beta^{-1}(n)$ is finite.
- (3) For $E \in \theta$, $M \in \Omega_\alpha$, and $A \in \mathcal{A}$, $M \cap f^{-1}(E) \not\subset A$. By (1), there exists $\Omega \in \beta N$ such that $\Omega \supset \bigcup_{\alpha < \omega_1} \Omega_\alpha$. By (2(a)), $L \in \Omega$ implies $L \supset M \in \Omega_\alpha$ for some $\alpha < \omega_1$. By (3), $f(\Omega) = \theta$ but $\Omega \sim \theta$. By (2(b)), Ω is a P -point (or a fixed ultrafilter but this is impossible since $f(\Omega) = \theta$).

So it will suffice to define the Ω_α .

Define $\Omega_0 = \{N\}$ and, for limit ordinals α , define $\Omega_\alpha = \bigcup_{\beta < \alpha} \Omega_\beta$. Then (1), (2), and (3) are trivially satisfied.

Suppose $\alpha = \beta + 1$ for some $\beta < \omega_1$. We find an $X \subset N$ satisfying (2(b)) such that, for all $L \in \Omega_\beta$ and $M = L \cap X$, (3) is satisfied. If for all $L \in \Omega_\beta$, (3) is satisfied with $M = L \cap X \cap S_\beta$ define $Y = X \cap S_\beta$. Otherwise since Ω_β and θ are closed under

finite intersection and $\mathcal{A}$ under finite union, (3) is satisfied with $M=L \cap X \cap (N-S_\beta)$; and in this case define $Y=X \cap (N-S_\beta)$. We then define $\Omega_\alpha=\Omega_{\alpha-1} \cup \{Y \cap L \mid L \in \Omega_{\alpha-1}\}$ and (1), (2), and (3) are all satisfied. We define X by cases.

Case 1. There is an $n \in N$ such that, for all $L \in \Omega_\beta$, $E \in \theta$, and $A \in \mathcal{A}$, $f_\beta^{-1}(n) \cap L \cap f^{-1}(E) \not\subset A$; then $X=f_\beta^{-1}(n)$ has the desired properties.

Case 2. For each $n \in N$ there exists $L_n \in \Omega_\beta$, $E_n \in \theta$ and $A_n \in \mathcal{A}$ such that $f_\beta^{-1}(n) \cap L_n \cap f^{-1}(E_n) \subset A_n$. Without loss of generality we assume that $A_n \subset A_{n+1}$, $E_n \supset E_{n+1}$, $L_n \supset L_{n+1}$ and, for $L \in \Omega_\beta$, there is an n such that $L \supset L_n$. For $j \in N$, define

$$D_j = \{e \in N \mid f^{-1}(e) \cap L_j - f_\beta^{-1}(1, 2, \dots, j) \text{ has more than } j \text{ terms}\}.$$

Observe that $D_j \in \theta$. Otherwise $D'=(N-D_j) \cap E_j \in \theta$. And by the definition of D_j there is a term A of $\mathcal{A}$ such that $f^{-1}(N-D_j) \cap L_j - f_\beta^{-1}(1, \dots, j) \subset A$. But by our assumption $f^{-1}(E_j) \cap L_j \cap f_\beta^{-1}(1, \dots, j) \subset A_j$. Hence $f^{-1}(D') \cap L_j \subset (A_j \cup A) \in \mathcal{A}$. But by (3) of our induction hypotheses, if $D' \in \theta$, $f^{-1}(D') \cap L_j$ is not a subset of any term of $\mathcal{A}$. Hence, since θ is a P -point, there is a $D \in \theta$ such that, for all $j \in N$, $D-D_j$ is finite. If $e \in D \cap D_1$, select $x_{e1} \in f^{-1}(e) \cap L_1 \cap f_\beta^{-1}(n)$ with n maximal. And for $j > 1$ and $j \in N$, if $e \in D \cap D_j$ select

$$x_{ej} \in f^{-1}(e) \cap L_j \cap f_\beta^{-1}(n) - (x_{e1}, x_{e2}, \dots, x_{e,j-1})$$

with n maximal. Let $X=\{x_{ej}\}$.

Fix $n \in N$ and let us show that $X \cap f_\beta^{-1}(n)$ is finite. By the definition of D_j and x_{ej} , if $e \in D_j$ and $x_{ej} \in f_\beta^{-1}(n)$, then $n > j$. Similarly, if $e \in D_n$ and x_{ej} is defined, $x_{ej} \in f_\beta^{-1}(m)$ for some $m > n$. So since x_{ej} is only defined for $e \in D$ and $D-D_n$ is finite, there are at most finitely many j and e such that $x_{ej} \in f_\beta^{-1}(n)$.

Now suppose $E \in \theta$ and $L \in \Omega_\beta$. Clearly $E \supset D \cap E$ and, for some i , $L \supset L_i$. For some $j > i$, let $A=\{x_{ek} \mid k < j\}$; then $A \in \mathcal{A}$. By (3) of our induction hypotheses $f^{-1}(D \cap E) \cap L_j \not\subset A$. But this implies that $\{e \in D \cap E \mid x_{ej} \text{ is defined}\} \neq \emptyset$ for any $j > i$. And this implies that $f^{-1}(D \cap E) \cap L_i \cap X$ is not a subset of any term of $\mathcal{A}$. Hence $f^{-1}(E) \cap L \cap X$ is not a subset of any term of $\mathcal{A}$ and (3) is satisfied with $M=L \cap X$.

IV. Let us describe a third partial order on βN which is between the other two. For Ω and θ in βN , let us say that Ω is *essentially greater than θ through f* if there is an $f \in F$ such that $f(\Omega)=\theta$ and, for $M \in \Omega$, $\{n \in N \mid f^{-1}(n) \cap M \text{ is infinite}\} \neq \emptyset$.

A. 1. Suppose $\Omega \sim \Lambda$ and $\theta \sim \phi$ and Ω is essentially greater than θ through f . Let π and p be permutations of N such that $\pi(\Lambda)=\Omega$ and $p(\theta)=\phi$. Then $p \circ f \circ \pi \in F$, $p \circ f \circ \pi(\Lambda)=\phi$ and, for $L \in \Lambda$, $\{n \in N \mid \pi^{-1} \circ f^{-1} \circ p^{-1}(n) \cap L \text{ is infinite}\} \neq \emptyset$. Hence Λ is essentially greater than ϕ .

2. For Ω and θ in βN , define $[\Omega] \sqsupseteq [\theta]$ if either $[\Omega]=[\theta]$ or Ω is essentially greater than θ . By 1, $\sqsupseteq$ is well defined. Clearly $\sqsupseteq$ is transitive and, by III A3, it is antisymmetric. Hence $\sqsupseteq$ is a partial order on the types in βN .

B. 1. Suppose Ω and θ belong to βN . Then $[\Omega] \geq [\theta]$ implies $[\Omega] \sqsupseteq [\theta]$ which implies $[\Omega] \supseteq [\theta]$.

2. By almost the same proofs, Theorems III B 2, 3, 5 and 6 are true with $\sqsubseteq$ replacing $\succeq$. However III B4 is false.

3. Observe that, if θ and Ω belong to βN and f and g to F and $f(\Omega)=g(\Omega)=\theta$, Ω may be essentially greater than θ through f but not through g . To see this choose any $\theta \in N^*$ and select a $g \in F$ such that $g^{-1}(n)$ has precisely n terms $x_{1n}, x_{2n}, \dots, x_{nn}$. Define $f \in F$ by $f^{-1}(i)=\{x_{in} \mid n \in N\}$. Recall that $x \in N$ implies x' is the fixed ultrafilter to which x belongs. Define $X_1=\{x'_{1n}\}_{n \in N} \in D$ and for $n > 1$ define $X_n \in D$ as $x'_{11}, x'_{22}, \dots, x'_{nn}, x'_{n,n+1}, x'_{n,n+2}, \dots$. For each $n \in N$, let $\rho_n = \theta_{x_n}$ and $X=\{\rho_n\}_{n \in N} \in D$ and $\Omega = \theta_X$. Then $g(\Omega)=\theta$ and $f(\Omega)=\theta$ and Ω is essentially greater than θ through f but not through g .

4. By definition $[\Omega]$ is minimal in N^* under $\sqsubseteq$ if and only if Ω is a P -point. Clearly N' is again minimal under $\sqsubseteq$ in βN .

5. The general character of $\sqsubseteq$ is more like that of $\succeq$ than that of $\geq$. However it has one nice property of $\geq$. If Ω and θ are of the same type in N^* , then the set of all predecessors of Ω under $\sqsubseteq$ is precisely the set of all predecessors of θ .

C. Together B 4 and 5 raised hope that the position in $\sqsubseteq$ of a type in βN might determine its type in N^* ; 2 destroys this hope. It also gives a constructive method of finding non- P -point types minimal in N^* under $\geq$. Using $\succeq$ and $\sqsubseteq$ together does not look useful as seen in 1.

1. Suppose Ω and θ are P -points in N^* . Then [CH] Ω and θ have the same type in N^* . And neither $[\Omega]$ nor $[\theta]$ has any predecessors under $\sqsubseteq$. But [CH] $[\Omega]$ and $[\theta]$ may be ordered by $\succeq$ or not ordered by $\succeq$. One can use sequences of P -points to show [CH] that there are two types in βN which (a) are of the same type in N^* , (b) have the same nonempty set of predecessors under $\sqsubseteq$, and (c) are comparable under $\succeq$; by the same method one can construct two types which satisfy (a), (b), and not (c).

2. There exist [CH] terms Ω , θ , and Δ of N^* such that $[\Delta]$ is minimal in N^* in $\succeq$, Δ is the only term of N^* essentially less than Ω and the only term of N^* essentially less than θ , but Ω and θ are not of the same type in N^* . In fact θ is a limit point of a countable discrete sequence of P -points, but Ω is not a limit point of any countable subset of N^* .

Proof. Choose $f \in F$ such that $f^{-1}(n)$ is infinite for each $n \in N$.

For $0 < \alpha < \omega_1$ and $n \in N$ select $\alpha_n \in \omega_1$ in such a way that, if α is not a limit ordinal, $\alpha_n = \alpha - 1$, and if α is a limit ordinal, $\{\alpha_n\}_{n \in N} = \{\beta \mid \beta < \alpha\}$.

By [CH], F and S can be indexed so $F = \{f_\alpha\}_{\alpha < \omega_1}$ and $S = \{S_\alpha\}_{\alpha < \omega_1}$. By a complicated induction on the countable ordinals, we define various subsets of N and points of N^* which in turn allow us to define Ω , θ , and $f(\theta)=f(\Omega)=\Delta$ with the desired properties.

For each countable ordinal α we wish to select

- (a) an infinite subset M_α of N ,
- (b) a countable ordinal $\alpha^* \geq \alpha$,
- (c) for each $n \in M_\alpha$ and $\beta \in \omega_1$, a subset $E_{\alpha n \beta}$ of $f^{-1}(n)$.

The following conditions are satisfied for all $n \in N$:

1. There is a P -point $p_{\alpha n} = \{U \subset N \mid \text{for some } \beta \in \omega_1, U \supset E_{\alpha n \beta}\}$ and $\delta < \beta < \omega_1$ implies that $E_{\alpha n \beta} - E_{\alpha n \delta}$ is finite and $E_{\alpha n \delta} - E_{\alpha n \beta}$ is infinite.
2. If $\gamma < \alpha$ and $E_{\alpha n 0} \cap E_{\gamma n \delta}$ is infinite, then $E_{\alpha n 0} - E_{\gamma n \delta}$ is finite.
3. If $\gamma < \alpha$, then $\gamma^* < \alpha^*$ and $E_{\gamma n(\alpha^*+1)} \cap E_{\alpha n 0}$ is finite, but there exists a $\delta < \alpha$ such that $E_{\alpha n 0} - E_{\delta n \alpha^*}$ is finite.
4. If $\alpha > 0$ and n is the i th term of M_α , then

$$n \in M_{\alpha_1} \cap M_{\alpha_2} \cap \cdots \cap M_{\alpha_i} \quad \text{and} \quad E_{\alpha_1 n 0} \cap E_{\alpha_2 n 0} \cap \cdots \cap E_{\alpha_i n 0} \cap E_{\alpha n 0}$$

is infinite.

In all cases, once $E_{\alpha n 0}$ has been chosen, choose $E_{\alpha n \beta}$ and $p_{\alpha n}$ in accordance with 1.

Let $M_0 = N$, $0^* = 0$ and, for all $n \in N$, $E_{0 n 0} = f^{-1}(n)$.

Assume our choices have been made for all $\gamma < \alpha$.

First suppose α is a limit ordinal. Choose $n_1 \in M_{\alpha_1}$. And, for all $i > 1$, choose $n_i \in M_{\alpha_1} \cap \cdots \cap M_{\alpha_i}$ such that $E_{\alpha_1 n_i 0} \cap \cdots \cap E_{\alpha_i n_i 0}$ is infinite and $n_i > n_{i-1}$. By 4, such n_i exist. Then let $M_\alpha = \{n_i\}_{i \in \mathbb{N}}$ and α^* be the limit of $\{\gamma^* \mid \gamma < \alpha\}$. Let $E_{\alpha n 0} = E_{\gamma n \alpha^*} - E_{\gamma n(\alpha^*+1)}$, where if $n = n_i$, γ is the largest of $\alpha_1, \dots, \alpha_i$, and otherwise $\gamma = 0$. One can check that 2, 3, and 4 are again satisfied.

Suppose $\alpha = \beta + 1$ and let g denote f_β .

Case 1. $X = \{n \in M_\beta \mid p_{\beta n} \notin g^{-1}(j) \text{ for any } j \in N\}$ is infinite. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in X$ and $j \in N$, $E_{\beta n \delta} \cap g^{-1}(j)$ is finite. Let $M = X$.

Case 2. X is finite and there exists an $i \in N$ and an infinite subset Z of M_β such that $n \in Z$ implies $p_{\beta n} \in g^{-1}(i)$. In this case there is a $\delta \in \omega_1$ such that, for all $n \in Z$, $E_{\beta n \delta} - g^{-1}(i)$ is finite. Let $M = Z$.

Case 3. Neither Case 1 nor 2 holds. Then there exist infinite subsets W of M_β and $\{a_j\}_{j \in N}$ of N such that $j < k$ in W implies $p_{\beta j} \in g^{-1}(a_j)$ and $p_{\beta k} \in g^{-1}(a_k)$ and $a_j < a_k$. In this case there exists a $\delta \in \omega_1$ such that, for all $n \in W$, $E_{\beta n \delta} - g^{-1}(a_n)$ is finite. Let $M = W$.

In all cases consider $g(M)$. If there is an infinite subset V of M and a $v \in N$ such that $g^{-1}(v) \supset V$, then let $M' = V$. Otherwise there is an infinite subset M' of M such that j and k belong to M' implies that $g(j) \neq g(k)$.

Choose $\alpha^* = \beta^* + \delta + 1$.

For some infinite subset M'' of M' , for all $n \in M''$, $Q = S_\beta \cap (E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)})$ is infinite or $Q = (N - S_\beta) \cap (E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)})$ is infinite. Let $M_\alpha = M''$.

If $n \notin M_\alpha$, let $E_{\alpha n 0} = E_{\beta n \alpha^*} - E_{\beta n(\alpha^*+1)}$.

In Case 1, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q - g^{-1}(1, \dots, n)$.

In Case 2, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q \cap g^{-1}(i)$.

In Case 3, if $n \in M_\alpha$, let $E_{\alpha n 0} = Q \cap g^{-1}(a_n)$.

It is easy to check that 2, 3 and 4 are again satisfied.

Let $\Omega = \{E \in S \mid \text{for some } \alpha \in \omega_1, E \supset \bigcup_{n \in M_\alpha} E_{\alpha n 0}\}$. By 4 and our selection of M'' and $E_{\alpha n 0}$, Ω is a free ultrafilter on N .

Let $f(\Omega) = \Delta$; observe that $\Delta = \{M \subset N \mid \text{for some } \alpha \in \omega_1, M \supset M_\alpha\}$.

Suppose $g \in F$. Then $g = f_\beta$ for some $\beta \in \omega_1$. If $\alpha = \beta + 1$ and $V = \bigcup_{n \in M_\alpha} E_{\alpha n 0}$, then $V \in \Omega$. In Case 1, for $j \in N$, $V \cap g^{-1}(j)$ is finite. In Case 2, $g(V) = i$ so $g(\Omega) = i'$. And in Case 3, $g(\Omega) \sim f(\Omega)$. So if $\phi \in \beta N$ and Ω is essentially greater than ϕ through g , either $[\phi] = N'$ or $[\phi] = [\Delta]$. This means that $[\Delta]$ is minimal in $\geq$ in N^* (hence Δ is a P -point) and $[\Delta]$ is the only type in N^* less than $[\Omega]$ in $\sqsubseteq$.

Select $X = \{x_n\}_{n \in N} \in D$ such that x_n is a P -point to which $f^{-1}(n)$ belongs. Let $\theta = \Delta_X$.

Suppose $g \in F$. Let $Y = \{n \in N \mid \text{for all } k \in N, g^{-1}(k) \notin x_n\}$. For $n \in Y$ we can select $L_n \in x_n$ such that $L_n \cap g^{-1}(1, \dots, n) = \emptyset$ and, for all $k \in N$, $L_n \cap g^{-1}(k)$ is finite. If $Y \in \Delta$, then $\bigcup_{n \in Y} L_n \in \theta$; so θ is not essentially greater than $g(\theta)$ through g if $Y \in \Delta$. If $Y \notin \Delta$, define $h \in F$ by $g^{-1}(h(n)) \in x_n$ for $n \in N - Y$ and $h(n) = 1$ for $n \in Y$. Then $h = f_\beta$ for some $\alpha - 1 = \beta \in \omega_1$ and $M_\alpha \cap (N - Y) \in \Delta$. By our definition of M' , either $h(M') = v$ for some $v \in N$ or h restricted to M' is one-to-one; but $M' \supset M_\alpha$. So $Y \notin \Delta$ implies $g(\theta) = v' \in N'$ or $g(\theta) \sim f(\theta)$. Hence, if θ is essentially greater than $g(\theta)$ through g and $g(\theta) \in N^*$, $[g(\theta)] = [f(\theta)]$. Thus $[\Delta]$ is the one type contained in N^* which is essentially less than θ or Ω .

Now we show that Ω is not a limit point of any countable subset of N^* ; one implication of this is that Ω and θ are not of the same type in N^* .

Suppose that $\{\rho_i\}_{i \in N}$ is a subset of $N^* - \{\Omega\}$. Let $A = \{i \in N \mid \text{for some } \alpha \in \omega_1 \text{ and } n \in N, \rho_i = p_{\alpha n}\}$. Let $B = \{i \in N \mid \text{for some } n \in N, f^{-1}(n) \in \rho_i \text{ but } \rho_i \neq p_{\alpha n} \text{ for any } \alpha \in \omega_1\}$. Let $C = \{i \in N \mid \text{for all } n \in N, f^{-1}(n) \notin \rho_i\}$. We find terms U, V , and W of Ω such that, for $i \in A$, $\rho_i \notin U$, for $i \in B$, $\rho_i \notin V$, and, for $i \in C$, $\rho_i \notin W$. Since $U \cap V \cap W \in \Omega$ and $A \cup B \cup C = N$, Ω is not a limit point of $\{\rho_i\}_{i \in N}$.

Choose α such that $\alpha > \gamma$ for all $\gamma \in \omega_1$ for which there are i and n in N such that $\rho_i = p_{\gamma n}$. Let $U = \bigcup_{n \in N} E_{\alpha n 0}$. If $\rho_i = p_{\gamma n}$, then $\gamma < \alpha$ and, by 1, $E_{\gamma n(\alpha^* + 1)} \in \rho_i$ and, by 3, $E_{\gamma n(\alpha^* + 1)} - E_{\alpha n 0} \in \rho_i$ and, by (c), $U \notin \rho_i$. Thus for $i \in A$, $U \in \Omega$ but $U \notin \rho_i$.

We want to choose a sequence $\{\beta^j\}_{j \in N} \subset \omega_1$, by induction. Let $\beta_1 = 0$. Suppose β^{j-1} has been selected. If $\gamma \in \omega_1$ and $n \in N$ and $i \in B$, by 1, there is a $\beta \in \omega_1$ such that $\beta \leq \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Thus we can select $\beta^j \in \omega_1$ such that $\beta^{j-1} < \beta^j$ and, for all $\gamma \leq \beta^{j-1}$ and $n \in N$ and $i \in B$, $\beta^j < \delta$ implies $E_{\gamma n \delta} \notin \rho_i$. Let α be the limit of $\{\beta^j\}_{j \in N}$ and let $V = \bigcup_{n \in N} E_{\alpha n 0} \in \Omega$. By 3, there is a $\gamma < \alpha$ such that $E_{\alpha n 0} - E_{\gamma n \alpha^*}$ is finite. So for $i \in B$, $E_{\alpha n 0} \notin \rho_i$. But for each $i \in B$ there is an $n \in N$ such that $f^{-1}(n) \in \rho_i$. Since $V \cap f^{-1}(n) = E_{\alpha n 0}$, $V \notin \rho_i$ for any $i \in B$.

For each $j \in N$, since $\Omega \neq \rho_j$, there is a $\delta^j \in \omega_1$ such that $\bigcup_{n \in M_{\delta^j}} E_{\delta^j n 0} \notin \rho_j$. Choose a limit ordinal α greater than δ^j for all $j \in N$. Suppose $j \in C$. There is an $i \in N$ such that $\delta^j = \alpha_i$. Let m be the i th term of M_α . Consider

$$W_j = \bigcup_{n \in M_\alpha} (E_{\alpha n 0} - E_{\alpha_i n 0}) - f^{-1}(1, \dots, m).$$

If $\bigcup_{n \in M_\alpha} E_{\alpha n 0} \in \rho_j$ then $W_j \in \rho_j$ for $j \in C$ implies $f^{-1}(1, \dots, m) \notin \rho_j$ and, by 4, $n \in M_\alpha$ and $n > m$ implies $n \in M_{\alpha_i}$ and $\delta^j = \alpha_i$ implies $\bigcup_{n \in M_{\alpha_i}} E_{\alpha_i n 0} \notin \rho_j$. Together 2 and 4 imply that, for $n \in M_\alpha$, $W_j \cap f^{-1}(n)$ is finite. For some $\beta \in \omega_1$, $\bigcup_{j \in C} W_j = S_\beta$;

since only finitely many W_j intersect $f^{-1}(n)$, $S_\beta \cap f^{-1}(n)$ is finite for all $n \in N$. But Ω is essentially greater than Δ through f , so $S_\beta \notin \Omega$. Thus $W = N - S_\beta \in \Omega$ but $W \notin \rho_j$ for any $j \in C$.

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